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ISOMETRIC APPROXIMATION

BY

P. ALESTALO

Matematiikan laitos, Teknillinen korkeakoulu
PL 1100, 02015 TKK, Finland
e-mail: palestal@falco.hut.fi

AND

D. A. TROTSENKO

Institut Matematiki SO RAN
Koptjuga prospekt 4, 630090 Novosibirsk, Russia
e-mail: trotsenk@math.nsc.ru

AND

J. VÄISÄLÄ

Matematiikan laitos, Helsingin yliopisto
PL 4, Yliopistonkatu 5, 00014 Helsinki, Finland
e-mail: jvaisala@cc.helsinki.fi

ABSTRACT

Suppose that $A \subset \mathbf{R}^n$ is a bounded set of diameter 1 and that $f: A \to l_2$ is a map satisfying the nearisometry condition $|x-y| - \varepsilon \le |fx - fy| \le |x-y| + \varepsilon$ with $\varepsilon \le 1$. Then there is an isometry $S: A \to l_2$ such that $|Sx - fx| \le c_n\sqrt{\varepsilon}$ for all x in A. If A satisfies a thickness condition and if $f: A \to \mathbf{R}^n$, then there is an isometry $S: \mathbf{R}^n \to \mathbf{R}^n$ with $|Sx - fx| \le c_n\varepsilon/q$, where q is a thickness parameter.

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1. Introduction

1.1. Let l_2 be the standard separable Hilbert space with inner product and norm written as $x \cdot y$ and $|x|$. We consider maps $f: A \to l_2$, where $A \subset l_2$, and f can change distances only slightly. The condition on f may be additive or multiplicative. More precisely, let $\varepsilon \geq 0$. We say that f is an ε-**nearisometry** if

$$|x - y| - \varepsilon \leq |fx - fy| \leq |x - y| + \varepsilon \tag{1.a}$$

for all $x, y \in A$, and that f is $(1 + \varepsilon)$-**bilipschitz** if

$$|x - y|/(1 + \varepsilon) \leq |fx - fy| \leq (1 + \varepsilon)|x - y| \tag{1.b}$$

for all $x, y \in A$.

The basic question is: how close is f to an actual isometry? For surjective ε-nearisometries $f: l_2 \to l_2$, Hyers and Ulam [HU] proved in 1945 that there is an isometry $S: l_2 \to l_2$ such that $|Sx - fx| \leq 10\varepsilon$ for all $x \in l_2$. The result was later extended to all Banach spaces; see [Ge], and the constant 10 has been reduced to 2; see [OŠ] and [Še].

Condition (1.a) is very strong for large distances, and the proofs of the results above make essential use of the behavior of f near the point at infinity.

In this paper, we consider bounded sets $A \subset l_2$, and the problem is essentially different. We show in 2.2 that if $A \subset \mathbf{R}^n$ is bounded and if $f: A \to l_2$ is an $\varepsilon d(A)$-nearisometry with $\varepsilon \leq 1$, then there is an isometry $S: \mathbf{R}^n \to l_2$ such that

$$|Sx - fx| \leq c_n \sqrt{\varepsilon} d(A)$$

for all $x \in A$. The result can be extended to the case where A lies in a narrow neighborhood of $\mathbf{R}^n$ in l_2; the constant c then depends also on the width of this neighborhood. The proofs are elementary but not short.

We do not know whether the result holds with a constant c independent of n or whether the result holds for all bounded subsets of l_2. On the other hand, the factor $\sqrt{\varepsilon}$ has the correct order of magnitude. We show in Section 3 that it can be replaced by ε if $f: A \to \mathbf{R}^n$, $A \subset \mathbf{R}^n$, and A is not close to any hyperplane.

For related earlier results, see [Jo], [Vä], [Fi] and [Tr].

A $(1 + \varepsilon)$-bilipschitz map $f: A \to l_2$ is an $\varepsilon d(A)$-nearisometry, and hence we obtain approximation results for bilipschitz maps. In Section 4 we apply the above results to approximate quasisymmetric maps by similarities.

The results of this paper can be applied to extension problems of bilipschitz and quasisymmetric maps.

1.2. NOTATION. We let $(e_1, e_2, \ldots)$ denote the standard basis of l_2, and the euclidean n-space $\mathbf{R}^n$ is identified with the linear subspace of l_2 generated by $e_1, \ldots, e_n$. We set $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_n \geq 0\}$. The distance between nonempty sets $A, B \subset l_2$ is written as $d(A, B)$. Furthermore, $d(A)$ is the diameter of A, and aff A is the affine subspace generated by A. For $x \in l_2$ and $k \in \mathbf{N}$ we set

$$x_{k*} = d(x, \mathbf{R}^{k-1}) = \Big(\sum_{i \geq k} x_i^2\Big)^{1/2}.$$

Then

$$x = \sum_{i=1}^{k-1} x_i e_i + x_{k*} e,$$

where $e = e(x, k)$ is a unit vector perpendicular to $\mathbf{R}^{k-1}$.

We let $B(x, r)$ and $\bar{B}(x, r)$ denote open and closed balls with center x and radius r, and we abbreviate $B(r) = B(0, r)$ $\bar{B}(r) = \bar{B}(0, r)$. If $A \subset l_2$ and f, g: $A \to l_2$ are maps, we write

$$\|f - g\|_A = \sup_{x \in A} |f(x) - g(x)|.$$

To simplify notation, we often omit parentheses writing $fx = f(x)$ etc.

1.3. SPECIAL CONVENTION. Given a map f: $A \to l_2$, we write $x' = f(x)$, $y' = f(y)$, etc. This convention is only applied if the map is denoted by f.

2. Nearisometries of arbitrary sets

2.1. SUMMARY. In this section we consider ε-nearisometries f: $A \to l_2$, where A is an arbitrary bounded set in $\mathbf{R}^n$. We show in Theorem 2.2 that f can be approximated by an isometry S: $\mathbf{R}^n \to l_2$ so that the error term is of the form $c\sqrt{\varepsilon}$. A more general result is given in 2.10, where A is allowed to lie in a narrow neighborhood of $\mathbf{R}^n$ in l_2.

To simplify the proof we assume that A is compact. This is not an essential restriction; see 2.11.

2.2. THEOREM: *Suppose that $A \subset \mathbf{R}^n$ is compact and that f: $A \to l_2$ is an $\varepsilon d(A)$-nearisometry with $\varepsilon \leq 1$. Then there is a surjective isometry S: $l_2 \to l_2$ such that $\|S - f\|_A \leq c_n \sqrt{\varepsilon} d(A)$, where c_n depends only on n. If $fA \subset \mathbf{R}^n$, we can choose S so that $S\mathbf{R}^n = \mathbf{R}^n$.*

The proof of 2.2 will be given in 2.8.

2.3. COROLLARY: *Suppose that $f\colon A \to l_2$ is as in 2.2. Then f has an extension to a δ-nearisometry $g\colon l_2 \to l_2$ with $\delta = c_n\sqrt{\varepsilon}d(A)$.*

Proof: Set $gx = Sx$ for $x \in l_2 \setminus A$, where S is given by 2.2. ∎

2.4. NORMALIZATION. We say that a finite sequence $\bar{a} = (a_0, \ldots, a_k)$ in l_2 is **normalized** if $a_0 = 0$ and if $a_i \in \mathbf{R}^i_+$ for $1 \le i \le k$. For each $\bar{a}$ there obviously is a surjective isometry $T\colon l_2 \to l_2$ such that $T\bar{a} = (Ta_0, \ldots, Ta_k)$ is normalized. If $A \subset l_2$ is a set containing a finite sequence $\bar{a}$ and if $f\colon A \to l_2$ is a map such that $f\bar{a}$ is normalized, we say that f is **normalized at** $\bar{a}$.

Given a compact set $A \subset l_2$, a finite sequence $\bar{u} = (u(0), \ldots, u(n))$ of $n+1$ points $u(j) \in A$ is said to be a **maximal n-sequence** in A if $|u(0) - u(1)| = d(A)$ and if for $2 \le k \le n$, the distance $d(u(k), \operatorname{aff}\{u(0), \ldots, u(k-1)\})$ is maximal in A. By compactness, the set A contains a maximal n-sequence for all $n \in \mathbf{N}$. If an n-sequence $\bar{u}$ is normalized and maximal in A, then $u(0) = 0$, $u(1) = d(A)e_1$, and $u(k) \in \mathbf{R}^k_+$ with $u(k)_k = \max\{x_{k*} : x \in A\}$; see 1.2 for notation.

If $n \in \mathbf{N}$, $A \subset l_2$ is compact and $f\colon A \to l_2$ is a map, then there are surjective isometries $T_1, T_2\colon l_2 \to l_2$ such that T_1A contains a normalized maximal n-sequence $\bar{u}$ and such that the map $T_2fT_1^{-1}\colon T_1A \to l_2$ is normalized at $\bar{u}$.

We start by estimating a nearisometry of a set of three points. Recall that we write $x' = fx$.

2.5. LEMMA: *Suppose that $A = \{0, e_1, x\} \subset \mathbf{R}^2$ with $d(A) = 1$. Let $\varepsilon \le 10^{-2}$, and let $f\colon A \to \mathbf{R}^2$ be an ε-nearisometry, normalized at $(0, e_1)$. Then*
(1) $|x_1 - x_1'| \le 3.03\varepsilon$,
(2) $|x_2^2 - {x_2'}^2| \le 6.2\varepsilon$,
(3) $|x_2 - x_2'| \le 6.2\varepsilon/x_2$ *if* $x_2 > 0$, $x_2' \ge 0$.

Proof: The proof makes use of the formula $2a \cdot b = |a|^2 + |b|^2 - |a-b|^2$. We have

$$\begin{aligned} 2|x_1 - x_1'| &= |2x \cdot e_1 - 2x' \cdot e_1| \le \big||x|^2 - |x'|^2\big| + \big||x - e_1|^2 - |x' - e_1|^2\big| \\ &= (|x| + |x'|)\,\big||x| - |x'|\big| + (|x - e_1| + |x' - e_1|)\,\big||x - e_1| - |x' - e_1|\big|. \end{aligned}$$

Since $|x'| \le |x| + \varepsilon \le 1 + \varepsilon$, the first term is at most $(2 + \varepsilon)\varepsilon$. To estimate the second term we write $fe_1 = \alpha e_1$ with $|\alpha - 1| \le \varepsilon$ and obtain $|x' - \alpha e_1| \le |x - e_1| + \varepsilon$, which yields

$$|x' - e_1| \le |x' - \alpha e_1| + |\alpha - 1| \le |x - e_1| + 2\varepsilon \le 1 + 2\varepsilon,$$

and similarly $|x' - e_1| \geq |x - e_1| - 2\varepsilon$. Thus

$$2|x_1 - x'_1| \leq (2+\varepsilon)\varepsilon + (2+2\varepsilon)2\varepsilon = 6\varepsilon + 5\varepsilon^2 \leq 6.05\varepsilon,$$

which gives (1).

To prove (2) we consider two cases.

CASE 1: $x_1 \leq 1/2$. Since $x_2^2 = |x|^2 - x_1^2$ and ${x'}_2^2 = |x'|^2 - {x'}_1^2$, and since $\varepsilon \leq 10^{-2}$, we get by (1)

$$\begin{aligned}|x_2^2 - {x'}_2^2| &\leq (|x| + |x'|)||x| - |x'|| + (|x_1| + |x'_1|)|x_1 - x'_1| \\ &\leq (2+\varepsilon)\varepsilon + (1 + 3.03\varepsilon)\cdot 3.03\varepsilon \leq 5.2\varepsilon.\end{aligned}$$

CASE 2: $x_1 \geq 1/2$. Now $x_2^2 = |x - e_1|^2 - (x_1 - 1)^2$ and ${x'}_2^2 = |x' - \alpha e_1|^2 - (x'_1 - \alpha)^2$, and thus

$$\begin{aligned}|x_2^2 - {x'}_2^2| \leq &(|x - e_1| + |x' - \alpha e_1|)\,||x - e_1| - |x' - \alpha e_1|| \\ &+ (|x_1 - 1| + |x'_1 - \alpha|)|x_1 - 1 - x'_1 + \alpha| \\ \leq &(2+\varepsilon)\varepsilon + (1/2 + |x'_1 - x_1| + |x_1 - 1| + |1 - \alpha|)(|x_1 - x'_1| + |\alpha - 1|) \\ \leq &2.01\varepsilon + (1 + 3.03\varepsilon + \varepsilon)(3.025\varepsilon + \varepsilon) \leq 6.2\varepsilon,\end{aligned}$$

where we used the estimate $|x_1 - x'_1| \leq 3.025\varepsilon$ from the proof of (1).

Finally, if $x_2 > 0$, $x'_2 \geq 0$, then $|x_2 + x'_2| \geq x_2$, and (3) follows from (2). ∎

2.6. AN INDUCTIVE STATEMENT. Theorem 2.2 will be proved by induction. For this purpose, let $n \in \mathbf{N}$ and let S_n be the following statement.

S_n: Let $A = \{0, e_1, u(2), \ldots, u(n), x\} \subset l_2$ be such that the n-sequence $\bar{u} = (0, e_1, u(2), \ldots, u(n))$ is normalized and maximal in A. Suppose that $f\colon A \to l_2$ is an ε-nearisometry with $\varepsilon \leq 10^{-2}$, normalized at $\bar{u}$. Then the following three estimates are true:

(i) If $\lambda > 0$ and if $u(n)_n \geq \lambda\sqrt{\varepsilon}$, then $|x_n - x'_n| \leq \varrho_n\varepsilon/u(n)_n$, where $\varrho_n = \varrho_n(\lambda)$.

(ii) If $\lambda > 0$ and if $u(n)_n \geq \lambda\sqrt{\varepsilon}$, then $|x_{(n+1)*}^2 - {x'}_{(n+1)*}^2| \leq \tau_n\varepsilon$, where $\tau_n = \tau_n(\lambda)$.

(iii) If $t \geq 0$ and if $x_{(n+1)*} \leq t\sqrt{\varepsilon}$, then $|x - x'| \leq \gamma_n\sqrt{\varepsilon}$, where $\gamma_n = \gamma_n(t)$.

The numbers ϱ_n, τ_n and γ_n are obtained from the following recursive formulas:

$$\varrho_1 = 3.03, \quad \tau_1 = 6.2, \quad \gamma_1(t)^2 = 0.1 + (t + \sqrt{t^2 + 6.2})^2,$$

$$\begin{aligned}\varrho_{n+1}(\lambda) &= 3.02 + \tau_n(\lambda)\sqrt{1 + \tau_n(\lambda)/\lambda^2} + \sum_{k=1}^{n} \varrho_k(\lambda)(2 + \varrho_k(\lambda)/\lambda^2), \\ \tau_{n+1}(\lambda) &= \tau_n(\lambda) + \varrho_{n+1}(\lambda)(2 + \varrho_{n+1}(\lambda)/\lambda^2), \\ \gamma_{n+1}(t) &= \min\{\max\{\gamma_n(\lambda), \beta_{n+1}(\lambda, t)\} : \lambda > 0\},\end{aligned}$$

where

$$\beta_{n+1}(\lambda,t)^2 = 0.1 + (t + \sqrt{t^2 + \tau_{n+1}(\lambda)})^2 + \lambda^{-2}\sum_{k=2}^{n+1} \varrho_k(\lambda)^2.$$

2.7. LEMMA: *Statement S_n is true for all $n \geq 1$.*

Proof: The lemma is proved by induction on n. Since $u(0) = 0$ and $u(1) = e_1$, we have $d(A) = 1$. Hence S_1(i) follows from 2.5(1); the condition involving λ is irrelevant. Estimate S_1(ii) follows from 2.5(2) with the aid of auxiliary rotations around span(e_1); again the condition with λ is irrelevant.

To prove S_1(iii), suppose that $t \geq 0$ and that $x_{2*} \leq t\sqrt{\varepsilon}$. From S_1(ii) we get

$$x'_{2*} \leq \sqrt{x_{2*}^2 + 6.2\varepsilon} \leq \sqrt{t^2 + 6.2}\sqrt{\varepsilon}.$$

Writing $x = x_1 e_1 + x_{2*} e$, $x' = x'_1 e_1 + x'_{2*} e'$ as in 1.2 we obtain

$$\begin{aligned}|x - x'|^2 = |x_1 - x'_1|^2 + |x_{2*}e - x'_{2*}e'|^2 &\leq 3.03^2 \cdot 10^{-2}\varepsilon + (x_{2*} + x'_{2*})^2 \\ &\leq (0.1 + (t + \sqrt{t^2 + 6.2})^2)\varepsilon,\end{aligned}$$

which proves S_1(iii).

Next assume that $n \geq 2$ and that S_k is true for $1 \leq k \leq n-1$. First observe that the functions $\varrho_{n+1}(\lambda)$, $\tau_{n+1}(\lambda)$ and $\beta_{n+1}(\lambda, t)$ are decreasing in λ and tend to infinity as $\lambda \to 0$. Moreover, $\beta_{n+1}(\lambda, t)$ is increasing in t and tends to infinity as $t \to \infty$. By induction we see that the definition of $\gamma_{n+1}(t)$ makes sense, and in fact, $\gamma_{n+1}(t) = \gamma_n(\lambda_n)$, where $\lambda_n = \lambda_n(t)$ is the unique solution of the equation $\gamma_n(\lambda_n) = \beta_{n+1}(\lambda_n, t)$. Moreover, $\gamma_n(t)$ is increasing in t.

Suppose that A, $\bar{u}$ and f are as in S_n. Writing $h_k = u(k)_k$, $h'_k = u(k)'_k$ we have $1 \geq h_1 \geq h_2 \geq \cdots \geq h_n$ and $h'_k \geq 0$. To prove S_n(i) we assume that $\lambda > 0$ and that $h_n \geq \lambda\sqrt{\varepsilon}$. We have

$$|x_n - x'_n|h_n \leq |x_n h_n - x'_n h'_n| + |x'_n||h_n - h'_n|.$$

Furthermore, the $(n-1)$-sequence $(0, e_1, u(2), \ldots, u(n-1))$ is normalized and maximal in A, and hence S_{n-1}(ii) yields $|h_n^2 - h'^2_n| \leq \tau_{n-1}\varepsilon$. Since $h_n > 0$ and $h'_n \geq 0$, this implies that

$$|h_n - h'_n| \leq \tau_{n-1}\varepsilon / h_n.$$

Since $x_{n*} \leq h_n$, condition S_{n-1}(ii) also gives

$$|x'_n| \leq x'_{n*} \leq \sqrt{x_{n*}^2 + \tau_{n-1}\varepsilon} \leq h_n\sqrt{1 + \tau_{n-1}/\lambda^2}.$$

On the other hand,

$$2\sum_{k=1}^{n} x_k u(n)_k = 2x \cdot u(n) = |x|^2 + |u(n)|^2 - |x-u(n)|^2.$$

This and the corresponding formula for $2x' \cdot u(n)'$ yield

$$2\left|\sum_{k=1}^{n}(x_k u(n)_k - x'_k u(n)'_k)\right| \le \left||x|^2 - |x'|^2\right| + \left||u(n)|^2 - |u(n)'|^2\right| + \left||x-u(n)|^2 - |x'-u(n)'|^2\right|.$$

The first term on the right is

$$(|x| + |x'|)\left||x| - |x'|\right| \le (1+1+\varepsilon)\varepsilon \le 2.01\varepsilon.$$

The other terms have the same upper bound, and hence

$$\left|\sum_{k=1}^{n}(x_k u(n)_k - x'_k u(n)'_k)\right| \le 3.015\varepsilon.$$

This implies that

$$|x_n h_n - x'_n h'_n| \le 3.015\varepsilon + \sum_{k=1}^{n-1}(|x_k - x'_k||u(n)_k| + |x'_k||u(n)_k - u(n)'_k|).$$

Since $|u(n)_k| \le h_k$, condition S_k(i) gives

$$|x_k - x'_k||u(n)_k| \le \varrho_k\varepsilon, \quad |u(n)_k - u(n)'_k| \le \varrho_k\varepsilon/h_k,$$
$$|x'_k| \le |x_k| + |x_k - x'_k| \le h_k + \varrho_k\varepsilon/h_k \le h_k(1+\varrho_k/\lambda^2)$$

for $1 \le k \le n-1$. Hence

$$|x_n h_n - x'_n h'_n| \le 3.015\varepsilon + \varepsilon\sum_{k=1}^{n-1}\varrho_k(2+\varrho_k/\lambda^2).$$

Combining the estimates yields

$$|x_n - x'_n| \le \frac{\varepsilon}{h_n}\left(3.02 + \tau_{n-1}\sqrt{1+\tau_{n-1}/\lambda^2} + \sum_{k=1}^{n-1}\varrho_k(2+\varrho_k/\lambda^2)\right) = \varepsilon\varrho_n/h_n,$$

and S_n(i) is proved.

We next prove S_n(ii). Let again $h_n \geq \lambda\sqrt{\varepsilon}$. We have

$$x^2_{(n+1)*} = x^2_{n*} - x^2_n, \quad x'^2_{(n+1)*} = x'^2_{n*} - x'^2_n.$$

Since $h_{n-1} \geq h_n \geq \lambda\sqrt{\varepsilon}$, we can apply S_{n-1}(ii) and obtain

$$|x^2_{(n+1)*} - x'^2_{(n+1)*}| \leq |x^2_{n*} - x'^2_{n*}| + |x^2_n - x'^2_n| \leq \tau_{n-1}\varepsilon + (|x_n| + |x'_n|)|x_n - x'_n|.$$

Here $|x_n| \leq h_n$ and

$$|x'_n| \leq |x_n| + |x_n - x'_n| \leq h_n + \varrho_n\varepsilon/h_n \leq h_n(1 + \varrho_n/\lambda^2).$$

Hence

$$|x^2_{(n+1)*} - x'^2_{(n+1)*}| \leq \left(\tau_{n-1} + \varrho_n(2 + \varrho_n/\lambda^2)\right)\varepsilon = \tau_n\varepsilon.$$

This completes the proof of S_n(ii).

To prove S_n(iii) assume that $t \geq 0$ and that $x_{(n+1)*} \leq t\sqrt{\varepsilon}$. Let $\lambda > 0$. We consider two cases.

CASE 1: $h_n \leq \lambda\sqrt{\varepsilon}$. Now $x_{n*} \leq \lambda\sqrt{\varepsilon}$, and S_{n-1}(iii) gives $|x - x'| \leq \gamma_{n-1}(\lambda)\sqrt{\varepsilon}$.

CASE 2: $h_n \geq \lambda\sqrt{\varepsilon}$. Write

$$x = \sum_{k=1}^{n} x_k e_k + x_{(n+1)*}e, \quad x' = \sum_{k=1}^{n} x'_k e_k + x'_{(n+1)*}e',$$

as in 1.2. Applying S_k(i) for $1 \leq k \leq n$ and S_n(ii) we obtain

$$\begin{aligned} |x - x'|^2 &= |x_1 - x'_1|^2 + \sum_{k=2}^{n} |x_k - x'_k|^2 + |x_{(n+1)*}e - x'_{(n+1)*}e'|^2 \\ &\leq (3.03\varepsilon)^2 + \sum_{k=2}^{n} (\varrho_k\varepsilon/h_k)^2 + (x_{(n+1)*} + x'_{(n+1)*})^2 \\ &\leq 0.1\varepsilon + \varepsilon\lambda^{-2}\sum_{k=2}^{n} \varrho_k^2 + (t + \sqrt{t^2 + \tau_n})^2\varepsilon = \beta_n(\lambda, t)^2\varepsilon. \end{aligned}$$

Since $\lambda > 0$ was arbitrary, S_n(iii) follows. ∎

2.8. PROOF OF 2.2. If $10^{-2} \leq \varepsilon \leq 1$, we fix a point $a \in A$ and choose an isometry $S\colon \mathbf{R}^n \to l_2$ with $Sa = fa$. Since $1 \leq 10\sqrt{\varepsilon}$, we get for each $x \in A$

$$|Sx - fx| \leq |Sx - Sa| + |fx - fa| \leq 2|x - a| + \varepsilon d(A) \leq 21\sqrt{\varepsilon}d(A).$$

Hence we may assume that $\varepsilon \leq 10^{-2}$.

Replacing f by the map $x \mapsto f(d(A)x)/d(A)$ we may assume that $d(A) = 1$. Moreover, we can use auxiliary isometries to normalize the situation so that A contains a normalized maximal n-sequence $\bar{u}$ and that f is normalized at $\bar{u}$. The theorem follows with $S = \text{id}$ and $c_n = \gamma_n(0)$ from the case $t = 0$ of statement S_n(iii) of 2.6. The condition $S\mathbf{R}^n \subset \mathbf{R}^n$ is clear from the proof. ∎

2.9. Numerical estimates. For $\varepsilon \leq 10^{-2}$, Theorem 2.2 is true with $c_1 = 2.51$, $c_2 = 8.73$, $c_3 = 18.8$.

We next give a more general version of 2.2 by allowing the set A to lie in a narrow neighborhood of $\mathbf{R}^n$.

2.10. Theorem: *Suppose that $n \in \mathbf{N}$, that F is an n-dimensional affine subspace of l_2, that $t \geq 0$, $0 \leq \varepsilon \leq 1$, and that $A \subset F + \bar{B}(t\sqrt{\varepsilon}d(A))$ is compact. Let $f\colon A \to l_2$ be an $\varepsilon d(A)$-nearisometry. Then there is a surjective isometry $S\colon l_2 \to l_2$ such that $\|S - f\|_A \leq \gamma_n(C_n t)\sqrt{\varepsilon}d(A)$, where the function γ_n is given by 2.7, and C_n depends only on n.*

Proof: We may again assume that $\varepsilon \leq 10^{-2}$ and $d(A) = 1$. Choose a maximal n-sequence $\bar{u} = (u(0), \ldots, u(n))$ for A. Let $T\colon l_2 \to l_2$ be a surjective isometry normalized at $\bar{u}$. Then TA lies in $\mathbf{R}^n + \bar{B}(C_n t\sqrt{\varepsilon})$ by Lemma 5.9 in the appendix. Hence we may assume that $A \subset \mathbf{R}^n + \bar{B}(C_n t\sqrt{\varepsilon})$ and that A has a normalized maximal n-sequence $\bar{u}$. By another auxiliary isometry we may assume that f is normalized at $\bar{u}$. Since $x_{(n+1)*} \leq C_n t\sqrt{\varepsilon}$ for all $x \in A$, the theorem follows from S_n(iii) of 2.6 with $S = \text{id}$. ∎

2.11. Remark. In 2.2 and 2.10, we assumed that A is compact in order that A contain a maximal sequence. However, the results are true for all bounded sets, and they are obtained by an elaboration of the proofs above, using approximate maximal sequences in the natural sense. Alternatively, we can in 2.2 use an extension of f to the closure $\bar{A}$.

Indeed, if A and fA lie in finite-dimensional subspaces, we can extend an ε-nearisometry $f\colon A \to l_2$ to an ε-nearisometry $g\colon \bar{A} \to l_2$ by setting $gx = \lim_{j\to\infty} fx_j$ for $x \in \bar{A} \setminus A$, where (x_j) is an arbitrary sequence in A such that $x_j \to x$ and such that the sequence (fx_j) converges.

In arbitrary metric spaces X and Y, one can extend an ε-nearisometry $f\colon A \to Y$, where $A \subset X$, to an ε'-nearisometry $g\colon \bar{A} \to Y$ for each $\varepsilon' > \varepsilon$ by setting $gx = fy$ for $x \in \bar{A} \setminus A$, where $y \in A$ is an arbitrary point with $|y - x| \leq (\varepsilon' - \varepsilon)/2$.

2.12. SHARPNESS. We next show that the bound $\|S-f\|_A \le c_n\sqrt{\varepsilon}d(A)$ in 2.2 has the correct order of magnitude. Indeed, we show that the theorem is not true for small ε if $c_n < 1/2$.

Set $A = \{0, e_1/2, e_1\} \subset \mathbf{R}$. Let $0 < h \le 1/2$, and define $f\colon A \to \mathbf{R}^2$ by $f(0) = 0$, $f(e_1) = e_1$, $f(e_1/2) = e_1/2 + he_2$. Then f is an ε-nearisometry with $\varepsilon = h^2$, since

$$\sqrt{1/4 + h^2} - 1/2 < h^2.$$

Assume that $S\colon A \to l_2$ is an isometry, and set $\alpha = \|S-f\|_A$. Then $S\mathbf{R}$ is a line meeting the three balls $\bar{B}(Sx, \alpha)$, $x \in A$. It follows that $\alpha \ge h/2 = \sqrt{\varepsilon}/2$. Consequently, for each $0 < \varepsilon \le 1/4$ there is a finite set $A \subset \mathbf{R}$ and an ε-nearisometry $f\colon A \to \mathbf{R}^2$ such that $d(A) = 1$ and $\|S-f\|_A \ge \frac{1}{2}\sqrt{\varepsilon}$ for every isometry $S\colon \mathbf{R} \to l_2$.

3. Nearisometries of thick sets

3.1. SUMMARY. In this section we consider nearisometries $f\colon A \to \mathbf{R}^n$, where A is compact in $\mathbf{R}^n$ and A is not very close to any hyperplane. We say that such a set is **thick** and give the precise definition for the thickness $\theta(A)$ of A in terms of projections onto lines. An alternative characterization for thickness in terms of heights of simplexes is considered in 3.13.

We show that ε-nearisometries of thick sets can be approximated by isometries with error term of the form $c\varepsilon$ instead of $c\sqrt{\varepsilon}$. Compactness can again be replaced by boundedness; see 2.11.

3.2. THICKNESS. For each unit vector $e \in S^{n-1}$ we define the projection $\pi_e\colon \mathbf{R}^n \to \mathbf{R}$ by $\pi_e x = x \cdot e$. Let $A \ne \varnothing$ be a bounded set in $\mathbf{R}^n$. The **thickness** of A is the number

$$\theta(A) = \inf\{d(\pi_e A) : e \in S^{n-1}\}.$$

Alternatively, $\theta(A)$ is the infimum of all $t > 0$ such that A lies between two parallel hyperplanes F, F' with $d(F, F') = t$. We have always $0 \le \theta(A) \le d(A)$.

3.3. THEOREM: *Suppose that $0 < q \le 1$ and that $A \subset \mathbf{R}^n$ is a compact set with $\theta(A) \ge qd(A)$. Let $f\colon A \to \mathbf{R}^n$ be an ε-nearisometry. Then there is an isometry $S\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\|S-f\|_A \le c_n\varepsilon/q$, where c_n depends only on n.*

The proof of 3.3 will be given in 3.7.

3.4. COROLLARY: *Suppose that q and $f\colon A \to \mathbf{R}^n$ are as in 3.3. Then f has an extension to a δ-nearisometry $g\colon \mathbf{R}^n \to \mathbf{R}^n$ with $\delta = c_n\varepsilon/q$.*

Proof: Set $gx = Sx$ for $x \in \mathbf{R}^n \setminus A$, where S is given by 3.3. ∎

3.5. AN INDUCTIVE STATEMENT. Theorem 3.3 will be proved by induction on n. For this purpose, let $n \geq 1$ be an integer and let T_n be the following statement:

T_n: Suppose that $A = \{0, e_1, u(2), \ldots, u(n), x\} \subset l_2$ is such that the n-sequence $\bar{u} = (0, e_1, u(2), \ldots, u(n))$ is normalized and maximal in A with $u(n)_n > 0$. Let $f\colon A \to l_2$ be an ε-nearisometry with $\varepsilon \leq 10^{-2} \wedge 4u(n)_n^2$, normalized at $\bar{u}$. Then

(i) $|x_n - x'_n| \leq \varrho_n \varepsilon / u(n)_n$,

(ii) $|x^2_{(n+1)*} - x'^2_{(n+1)*}| \leq \tau_n \varepsilon$.

The numbers ϱ_n and τ_n depend only on n, and they are obtained from the following recursive formulas:

$$\varrho_1 = 3.03, \quad \tau_1 = 6.2,$$
$$\varrho_n = 3.02 + \tau_{n-1}\sqrt{1 + \tau_{n-1}} + 2\sum_{k=1}^{n-1} \varrho_k(1 + 2\varrho_k),$$
$$\tau_n = \tau_{n-1} + 2\varrho_n(1 + 2\varrho_n).$$

3.6. LEMMA: *Statement T_n is true for all $n \geq 1$.*

Proof: The case $n = 1$ follows from 2.5 with auxiliary rotations. Assume that $n \geq 2$ and that T_k is true for $1 \leq k \leq n-1$. Let $f\colon A \to l_2$ be as in T_n. Write $h_k = u(k)_k$ and $h'_k = u(k)'_k$. From the maximality and normalization conditions it follows that

$$1 = h_1 \geq \cdots \geq h_n > 0,$$

and that

$$|x_k| \leq x_{k*} \leq u(k)_{k*} = h_k, \quad h'_k = u(k)'_{k*} \tag{3.a}$$

for all $1 \leq k \leq n$. Moreover,

$$\varepsilon \leq 4h_n^2 \leq 4h_k^2 \tag{3.b}$$

for all $1 \leq k \leq n$.

To prove T_n(i) we first observe that

$$h_n|x_n - x'_n| \leq |x_n h_n - x'_n h'_n| + |x'_n||h_n - h'_n|. \tag{3.c}$$

As in the proof of 2.7, we have $|x \cdot u(n) - x' \cdot u(n)'| \leq 3.015\varepsilon$. Since $u(n), u(n)' \in \mathbf{R}^n$, we obtain

$$\begin{aligned}|x_n h_n - x'_n h'_n| &\leq |x \cdot u(n) - x' \cdot u(n)'| + \sum_{k=1}^{n-1} |x_k u(n)_k - x'_k u(n)'_k| \\ &\leq 3.015\varepsilon + \sum_{k=1}^{n-1} (|x_k - x'_k||u(n)_k| + |x'_k||u(n)_k - u(n)'_k|).\end{aligned}$$

From (3.b) it follows that T_k holds for the restrictions of f to the sets

$$A_k = \{0, e_1, u(2), \dots, u(k), x\}, \quad B_k = \{0, e_1, u(2), \dots, u(k), u(n)\}$$

for all $1 \leq k \leq n-1$. From T_k(i) for $f|A_k$ we get for these k

$$\begin{aligned}&|x_k - x'_k| \leq \varrho_k \varepsilon / h_k, \\ &|x'_k| \leq |x_k| + |x_k - x'_k| \leq h_k + \varrho_k \varepsilon / h_k \leq (1 + 4\varrho_k) h_k,\end{aligned}$$

since $\varepsilon \leq 4h_n^2 \leq 4h_k^2$.

Furthermore, T_k(i) for $f|B_k$ gives, together with (3.a),

$$|u(n)_k - u(n)'_k| \leq \varrho_k \varepsilon / h_k.$$

Since $|u(n)_k| \leq h_k$ by the maximality of $\bar{u}$, these estimates yield

$$|x_n h_n - x'_n h'_n| \leq 3.015\varepsilon + 2\varepsilon \sum_{k=1}^{n-1} \varrho_k (1 + 2\varrho_k). \tag{3.d}$$

Applying T_{n-1}(ii) to $f|A_{n-1}$ we obtain

$$|x'_n| \leq |x'_{n*}| \leq \sqrt{x_{n*}^2 + \tau_{n-1}\varepsilon} \leq h_n \sqrt{1 + \tau_{n-1}}, \tag{3.e}$$

and T_{n-1}(ii) for $f|B_{n-1}$ gives

$$|h_n - h'_n| = \frac{|h_n^2 - h'^2_n|}{h_n + h'_n} \leq \frac{\tau_{n-1}\varepsilon}{h_n}, \tag{3.f}$$

in view of (3.a). Condition T_n(i) follows now from the estimates (3.c)–(3.f).

To prove T_n(ii) we first observe that

$$|x_{(n+1)*}^2 - x'^2_{(n+1)*}| \leq |x_{n*}^2 - x'^2_{n*}| + |x_n^2 - x'^2_n|.$$

Applying T_{n-1}(ii) to $f|A_{n-1}$ we get

$$|x_{n*}^2 - x'^2_{n*}| \leq \tau_{n-1}\varepsilon.$$

Furthermore, T_n(i) implies that

$$|x'_n| \le |x_n| + |x_n - x'_n| \le h_n + \varrho_n\varepsilon/h_n \le h_n(1 + 4\varrho_n).$$

Together with T_n(i) and (3.a), this implies that

$$\begin{aligned}|x_n^2 - {x'}_n^2| &\le (|x_n| + |x'_n|)|x_n - x'_n| \le (h_n + h_n(1 + 4\varrho_n))\varrho_n\varepsilon/h_n \\ &= 2\varrho_n(1 + 2\varrho_n)\varepsilon.\end{aligned}$$

These estimates yield T_n(ii). ∎

3.7. Proof of Theorem 3.3. We may assume that $d(A) = 1$. Choose a maximal n-sequence $\bar{u} = (u(0), \ldots, u(n))$ in A. Performing auxiliary isometries we may assume that $\bar{u}$ is normalized and that f is normalized at $\bar{u}$. Then $u(0) = 0$ and $u(1) = e_1$. Setting $h_k = u(k)_k$ we have $1 = h_1 \ge \cdots \ge h_n$. Moreover, since $|x_n| \le h_n$ for all $x \in A$, we have $2h_n \ge \theta(A) \ge q$.

If $q^2 \le \varepsilon \le 1$, we apply 2.2 to get an isometry S with $\|S - f\|_A \le c_n\sqrt{\varepsilon} = c_n\varepsilon/\sqrt{\varepsilon} \le c_n\varepsilon/q$.

If $\varepsilon \ge 10^{-2}$, then

$$|x - x'| \le |x| + |x'| \le 2 + \varepsilon \le 201\varepsilon \le 201\varepsilon/q,$$

and hence $S = \mathrm{id}$ is the desired approximation of f.

Finally, let $\varepsilon \le 10^{-2} \wedge q^2$. Then $\varepsilon \le q^2 \le 4h_k^2$ for all $1 \le k \le n$.

We show that $S = \mathrm{id}$ is the desired isometric approximation of f. Let $x \in A$. For $1 \le k \le n$, we can apply condition T_k(i) of 3.5 to $f|\{0, e_1, u(2), \ldots, u(k), x\}$ and obtain

$$|x_k - x'_k| \le \varrho_k\varepsilon/h_k \le 2\varrho_k\varepsilon/q.$$

Consequently,

$$|x - x'|^2 = \sum_{k=1}^{n} |x_k - x'_k|^2 \le 4\varepsilon^2 q^{-2} \sum_{k=1}^{n} \varrho_k^2,$$

which gives $|x - x'| \le c_n\varepsilon/q$ with $c_n^2 = 4\sum_{k=1}^{n} \varrho_k^2$. ∎

3.8. Numerical estimates. For $\varepsilon \le q^2 \wedge 10^{-2}$, Theorem 3.3 is true with $c_1 = 6.06$, $c_2 = 126$, $c_3 = 4 \cdot 10^6$.

3.9. Sharpness. We show that the bound $\|S - f\|_A \le c_n\varepsilon/q$ has the correct order of magnitude as a function of q. For $0 < q \le 1/2$ let A be the set $\{0, u, e_1\}$, where $u = e_1/2 + qe_2$. Then $\theta(A) = q$ and $d(A) = 1$. For $0 < h \le q/3$ define a map $f\colon A \to \mathbf{R}^2$ by $f(0) = 0$, $f(e_1) = e_1$, $f(u) = u + he_2$. Then f is an ε-nearisometry with $\varepsilon = 7qh/3$, since

$$\sqrt{1/4+(q+h)^2} - \sqrt{1/4+q^2} \le 2qh + h^2 \le 2qh + qh/3 = \varepsilon.$$

If $S\colon \mathbf{R}^2 \to \mathbf{R}^2$ is an isometry with $\|S-f\|_A \le \alpha$, then $q+h-2\alpha \le q$, and hence $\alpha \ge h/2 > \varepsilon/6q$. Consequently, for every $0 < q \le 1/2$ and $0 < \varepsilon \le 7q^2/9$ there is a set $A \subset \mathbf{R}^2$ and an ε-nearisometry $f\colon A \to \mathbf{R}^2$ such that $\theta(A) = q$, $d(A) = 1$, and $\|S - f\|_A \ge \varepsilon/6q$ for every isometry $S\colon \mathbf{R}^2 \to \mathbf{R}^2$.

3.10. John's method. The proofs of 2.2 and 3.3 were, unfortunately, technically complicated. On the other hand, F. John [Jo] has given an elegant proof for isometric approximation of locally $(1+\varepsilon)$-bilipschitz maps $f\colon G \to \mathbf{R}^n$, where the domain $G \subset \mathbf{R}^n$ belongs to a class later called John domains. For example, G may be a convex domain with $B(x_0, r) \subset G \subset B(x_0, cr)$ with a given $c \ge 1$.

We next show that John's ideas can be applied to approximate a nearisometry $f\colon A \to \mathbf{R}^n$ provided that the set $A \subset \mathbf{R}^n$ possesses "enough orthogonality". The proof is based on the following fact; see [BL, page 349].

3.11. Lemma: *If $A\colon \mathbf{R}^n \to \mathbf{R}^n$ is a linear map such that $|Ae_i \cdot Ae_j - e_i \cdot e_j| \le t$ for all i and j, then there is a linear orthogonal map T such that $|A - T| \le nt$, where $|\cdot|$ is the operator norm.*

3.12. Theorem: *Suppose that $n \ge 2$, that $A \subset \bar{B}^n(x_0, R)$, and that A contains points $x_0, x_0+ru_1, \ldots, x_0+ru_n$, where $0 < r \le R$, and the vectors $u_1, \ldots, u_n$ are orthonormal. Let $f\colon A \to \mathbf{R}^n$ be an ε-nearisometry. Then there is an isometry $S\colon A \to \mathbf{R}^n$ such that $Sx_0 = fx_0$ and $\|S - f\|_A \le 10n^{3/2}R\varepsilon/r$.*

Proof: We may normalize the situation so that $x_0 = 0 = fx_0$, $r = 1$, $u_i = e_i$. If $\varepsilon \ge 1/4$, we can choose $S = \mathrm{id}$, since

$$|x - fx| \le |x| + |fx| \le 2R + \varepsilon \le 9R\varepsilon$$

for all $x \in A$. Assume that $\varepsilon \le 1/4$. Applying again the basic formula $2a \cdot b = |a|^2 + |b|^2 - |a-b|^2$ we get for all $x, y \in A$

$$\begin{aligned} 2|fx \cdot fy - x \cdot y| &\le (|x| + |fx|)\varepsilon + (|y| + |fy|)\varepsilon + (|x-y| + |fx - fy|)\varepsilon \\ &\le (2|x| + 1/4)\varepsilon + (2|y| + 1/4)\varepsilon + (2|x-y| + 1/4)\varepsilon, \end{aligned}$$

and hence

$$|fx \cdot fy - x \cdot y| \le (|x| + |y| + |x-y| + 3/8)\varepsilon. \tag{3.g}$$

Let $L\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear map with $Le_i = fe_i$, $1 \le i \le n$. Then (3.g) implies that

$$|Le_i \cdot Le_j - e_i \cdot e_j| \le (1 + 1 + \sqrt{2} + 3/8)\varepsilon \le 3.8\varepsilon.$$

By 3.11, there is an orthogonal map $T\colon \mathbf{R}^n \to \mathbf{R}^n$ with $|T - L| \le 3.8n\varepsilon$. By (3.g) we get

$$\begin{aligned}|(fx - Tx) \cdot Te_i| &\le |fx|\,|Te_i - Le_i| + |fx \cdot fe_i - x \cdot e_i| \\ &\le 3.8n\varepsilon(R + 1/4) + (R + 1 + R + 1 + 3/8)\varepsilon \le 9.2nR\varepsilon\end{aligned}$$

for all $x \in A$ and $1 \le i \le n$. Since the vectors Te_i are orthonormal, this implies $|fx - Tx| \le 9.2n^{3/2}R\varepsilon$. ∎

3.13. Simplicial thickness. Let Δ be a k-simplex in l_2, and let Δ^0 be the set of vertices of Δ. We let $b(\Delta)$ denote the smallest height of Δ, that is, the smallest distance between a vertex of Δ and the affine subspace spanned by the opposite face.

For a bounded set $\varnothing \ne A \subset \mathbf{R}^n$, the number

$$\theta_S(A) = \sup\{b(\Delta) : \Delta \text{ is an } n\text{-simplex, } \Delta^0 \subset A\}$$

is the **simplicial thickness** of A. A localized version of this concept has been used in [Vä] and [VVW].

It is rather obvious that the numbers $\theta(A)$ and $\theta_S(A)$ cannot be too different from each other, and in the next result we show that the ratio $\theta_S(A)/\theta(A)$ lies between two positive bounds. It is perhaps not so obvious that the upper bound must depend on the dimension n, but if A is a regular unit n-simplex, then $\theta_S(A) > 1/\sqrt{2}$ while $\theta(A)$ is roughly $\sqrt{2/n}$.

3.14. Theorem: *If $A \ne \varnothing$ is a bounded set in $\mathbf{R}^n$, then*

$$\theta(A)/2 \le \theta_S(A) \le (n + 1)\theta(A)/2.$$

Proof: We may assume that A is compact. Choose an n-simplex Δ with $\Delta^0 \subset A$ such that the volume of Δ is maximal. Write $b(\Delta) = d(v, E)$ where v is a vertex of Δ and E is the affine subspace spanned by the opposite face. Now A lies in $E + \bar{B}(b(\Delta))$, since otherwise we could find a simplex with larger volume. This proves the first inequality.

The second inequality follows from Theorem 5.3 in the appendix. For an even n we have a better estimate $\theta_S(A) \le n\theta(A)/2$. ∎

4. Quasisymmetric and nearsymmetric maps

In this section we consider the approximation of maps by similarities instead of isometries.

4.1. QUASISYMMETRY. We first recall the concept of a quasisymmetric map. In this section, we let X and Y denote metric spaces, and the distance between points a and b in each space is written as $|a - b|$. A **proper triple** in X is a triple $T = (x; y, z)$ of points in X such that $y \neq x \neq z$. The **ratio** of a proper triple $T = (x; y, z)$ is the number

$$|T| = \frac{|x - y|}{|x - z|}.$$

An injective map $f\colon X \to Y$ maps each proper triple $T = (x; y, z)$ in X to the proper triple

$$T' = fT = (fx; fy, fz)$$

in Y.

Let $\eta\colon [0, \infty[\to [0, \infty[$ be a homeomorphism. An injective map $f\colon X \to Y$ is η-**quasisymmetric** if $|T'| \leq \eta(|T|)$ for every proper triple T in X. For $0 \leq s \leq 1$, a map $f\colon X \to Y$ is s-**quasisymmetric** if f is η-quasisymmetric for some η and if $|T'| \leq |T| + s$ for all proper triples in X with $|T| \leq 1/s$. For $s = 0$, this means that $|T'| = |T|$ for all T, and f is a **similarity**.

4.2. NEARSYMMETRY. Let $0 \leq s \leq 1$. We say that an injective map $f\colon X \to Y$ is s-**nearsymmetric** if $|T'| \leq |T| + s$ for all proper triples T in X with $|T| \leq 1/s$. An s-quasisymmetric map is trivially s-nearsymmetric, but an s-nearsymmetric map need not be η-quasisymmetric for any η. However, this is the case if the space X has suitable connectedness properties; see [TV, 3.10] and [Vä, 2.3].

We show that suitably normalized nearsymmetric maps are nearisometries, and we apply this fact to prove that nearsymmetric (and hence quasisymmetric) maps can be approximated by similarities.

4.3. LEMMA: *Suppose that $f\colon X \to Y$ is s-nearsymmetric and that T is a proper triple in X with $|T| \leq r$ where $r \geq 1$. Then*

$$|T'| \geq |T| - r^2 s.$$

Proof: We may assume that $s > 0$. Let $T = (x; y, z)$. Then $U = (x; z, y)$ is a proper triple. If $|T| \leq rs$, the lemma holds trivially. If $|T| \geq rs$, then $|U| = |T|^{-1} \leq (rs)^{-1} \leq s^{-1}$, and hence

$$|T'|^{-1} = |U'| \leq |U| + s = |T|^{-1} + s.$$

This implies that $|T'| \geq |T| - s'$ where

$$s' = \frac{|T|^2 s}{1 + s|T|} \leq |T|^2 s \leq r^2 s. \qquad \blacksquare$$

4.4. THEOREM: *Suppose that $0 \leq s \leq 1/2$ and that $f\colon X \to Y$ is s-nearsymmetric. Suppose also that there are points $a, b \in X$ with $|a - b| = 1 = d(X)$ and $|fa - fb| = 1$. Then f is a $6s$-nearisometry.*

Proof: Let $x, y \in A$, $x \neq y$. Since $|x - a| + |x - b| \geq |a - b| = 1$, we may assume that $|x - a| \geq 1/2$. Then $T_1 = (x; y, a)$ and $T_2 = (a; x, b)$ are proper triples in X, and

$$|x - y| = |T_1||T_2|, \quad |fx - fy| = |T_1'||T_2'|, \quad |T_1| \leq 2, \quad |T_2| \leq 1.$$

Since $s \leq 1/2$, it follows from the nearsymmetry of f that

$$\begin{aligned} |fx - fy| &= |T_1'||T_2'| \leq (|T_1| + s)(|T_2| + s) \\ &= |x - y| + |T_1|s + |T_2|s + s^2 \\ &< |x - y| + 4s. \end{aligned}$$

Since $|T_1| \leq 2$ and $|T_2| \leq 1$, Lemma 4.3 gives

$$|T_1'| \geq |T_1| - 4s, \ |T_2'| \geq |T_2| - s.$$

Here $|T_2| = |x - a| \geq 1/2 \geq s$, and hence

$$\begin{aligned} |fx - fy| &= |T_1'||T_2'| \geq (|T_1| - 4s)(|T_2| - s) \\ &= |x - y| - 4s|x - a| - s|x - y|/|x - a| + 4s^2 \geq |x - y| - 6s. \end{aligned}$$

Hence f is a $6s$-nearisometry. $\blacksquare$

4.5. THEOREM: *Suppose that $A \subset \mathbf{R}^n$ is compact and that $f\colon A \to l_2$ is s-nearsymmetric with $s \leq 1/2$. Then there is a surjective similarity $S\colon l_2 \to l_2$ such that $\|S \circ f - \mathrm{id}\,\|_A \leq c_n\sqrt{s}d(A)$, where c_n depends only on n. If $fA \subset \mathbf{R}^n$, we can choose S so that $S\mathbf{R}^n = \mathbf{R}^n$.*

Proof: Choose points $a, b \in A$ with $|a - b| = d(A)$ and set $A_0 = A/d(A)$, $M = |fa - fb|$. The map $g\colon A_0 \to l_2$, defined by $gx = f(d(A)x)/M$, is s-nearsymmetric. From Theorem 4.4 it follows that g is a $6s$-nearisometry. By Theorem 2.2, there is a surjective isometry $S_0\colon l_2 \to l_2$ such that $\|S_0 - g\|_{A_0} \leq c_n\sqrt{6s}d(A)$. Setting $Sx = d(A)S_0^{-1}(x/M)$ we obtain a surjective similarity $S\colon l_2 \to l_2$ with $\|S \circ f - \mathrm{id}\,\|_A \leq c_n\sqrt{6s}d(A)$. $\blacksquare$

Similarly, Theorems 3.3 and 4.4 give the following result:

4.6. THEOREM: *Suppose that $0 < q \leq 1$ and that $A \subset \mathbf{R}^n$ is a compact set with $\theta(A) \geq qd(A)$. Let $f\colon A \to \mathbf{R}^n$ be s-nearsymmetric with $s \leq 1/2$. Then there is a similarity $S\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\|S \circ f - \mathrm{id}\,\|_A \leq c_n s d(A)/q$, where c_n depends only on n.*

5. Appendix: Simplexes

In this appendix we prove some elementary results on simplexes, needed in the proofs of 2.10 and 3.14. However, they are not needed in the proofs of the main results 2.2 and 3.3.

5.1. LEMMA: *Suppose that $\Delta \subset \mathbf{R}^n$ is an n-simplex with vertices $a_0, \ldots, a_p$, $b_0, \ldots, b_q$, where $a_j \in \mathbf{R}^{n-1}$ and $b_j \in \mathbf{R}^{n-1} + e_n$. Set*

$$E = \mathrm{aff}\{a_0, \ldots, a_p\}, \quad F = \mathrm{aff}\{b_0, \ldots, b_q\}.$$

Then $E \cap (F - e_n) \neq \varnothing$.

Proof: We may assume that $a_0 = 0$. Set $y = b_0 - e_n$ and $k_j = b_j - b_0$, $1 \leq j \leq q$. Then

$$E = \mathrm{span}\{a_1, \ldots, a_p\}, \quad F = b_0 + \mathrm{span}\{k_1, \ldots, k_q\}.$$

Since Δ is an n-simplex, the vectors $a_1, \ldots, a_p, b_0, \ldots, b_q$ form a basis of $\mathbf{R}^n$. Hence we can write

$$\begin{aligned} y &= s_1 a_1 + \cdots + s_p a_p + t_0 b_0 + \cdots + t_q b_q \\ &= s_1 a_1 + \cdots + s_p a_p + (t_0 + \cdots + t_q) b_0 + t_1 k_1 + \ldots + t_q k_q. \end{aligned}$$

Since the vectors y, a_j, k_j lie in $\mathbf{R}^{n-1}$ while $b_0 \notin \mathbf{R}^{n-1}$, this implies

$$s_1 a_1 + \cdots + s_p a_p = y - t_1 k_1 - \cdots - t_q k_q \in E \cap (F - e_n). \qquad \blacksquare$$

We recall some notation from Section 3. For $e \in S^{n-1}$, the projection $\pi_e\colon \mathbf{R}^n \to \mathbf{R}$ is defined by $\pi_e x = x \cdot e$. The thickness of a bounded set $A \subset \mathbf{R}^n$ is $\theta(A) = \min d(\pi_e A)$ over $e \in S^{n-1}$. The smallest height of a simplex Δ is $b(\Delta)$, and Δ^0 is the set of vertices of Δ.

5.2. LEMMA: *Let $\Delta \subset \mathbf{R}^n$ be an n-simplex, and let $z \in S^{n-1}$ be such that $d(\pi_z \Delta) = \theta(\Delta)$. Then $\pi_z \Delta$ is an interval $[a, b]$ such that $\pi_z \Delta^0 = \{a, b\}$.*

Proof: Assume that the lemma is false. Then Δ has a vertex v such that $\pi_z v$ is an interior point of the interval $J = \pi_z \Delta$. Let Δ_1 be the $(n-1)$-face of Δ

opposite to v. Then $J = \pi_z \Delta_1$. There is $\delta > 0$ such that $\pi_e \Delta_1 = \pi_e \Delta$ whenever $e \in S^{n-1}$ and $|e - z| < \delta$. We may assume that $0 \in E = \operatorname{aff} \Delta_1$. We can write $z = e_1 \cos\alpha + e_2 \sin\alpha$, where $0 \le \alpha < \pi/2$, and e_1, e_2 are unit vectors such that $e_1 \in E$ and $e_2 \perp E$. For $0 \le \varphi \le \pi/2$, we set $e_\varphi = e_1 \cos\varphi + e_2 \sin\varphi$ and $g(\varphi) = d(\pi_{e_\varphi} \Delta_1)$. Then $g(\varphi) = g(0)\cos\varphi$, and hence g is strictly decreasing. Choosing $\varphi \in \,]\alpha, \pi/2]$ with $|e_\varphi - e_\alpha| < \delta$ we have $g(e_\varphi) < g(e_\alpha) = g(z)$, and hence $d(\pi_z \Delta)$ is not minimal. ∎

5.3. THEOREM: *Let $\Delta \subset \mathbf{R}^n$ be an n-simplex. Then $b(\Delta) \le (n+1)\theta(\Delta)/2$. If n is even, then $b(\Delta) \le n\theta(\Delta)/2$.*

Proof: Write $n = 2k - 1$ if n is odd and $n = 2k$ if n is even.

We may assume that $\theta(\Delta) = 1$. Choose a vector $z \in S^{n-1}$ such that $d(\pi_z \Delta) = \theta(\Delta)$. We may assume that $z = e_n$ and that $0 \le x_n \le 1$ for all $x \in \Delta$. We must show that $b(\Delta) \le k$.

By 5.2 we have $\Delta^0 \subset \mathbf{R}^{n-1} \cup (\mathbf{R}^{n-1} + e_n)$. Write $\Delta^0 = \{a_0, \dots, a_p, b_0, \dots, b_q\}$ where $a_j \in \mathbf{R}^{n-1}$, $b_j \in \mathbf{R}^{n-1} + e_n$. Then $p + q = n - 1$. We may assume that $p \le q$, and hence $p \le k - 1$. If $p = 0$, then $b(\Delta) \le d(a_0, \mathbf{R}^{n-1} + e_n) = 1$. Assume that $p \ge 1$. Set

$$E = \operatorname{aff}\{a_0, \dots, a_p\}, \quad F = \operatorname{aff}\{b_0, \dots, b_q\}.$$

By 5.1 there is a point $x_0 \in E \cap (F - e_n)$. Since $x_0 \in E$, we can write

$$x_0 = \sum_{i=0}^{p} t_i a_i, \quad \text{where } \sum_{i=0}^{p} t_i = 1.$$

We may assume that $t_0 \ge 1/(p+1)$. Set

$$a = \frac{1}{1 - t_0} \sum_{i=1}^{p} t_i a_i.$$

Then $a \in \operatorname{aff}\{a_1, \dots, a_p\}$ and $x_0 = (1 - t_0)a + t_0 a_0$.

Let T be the affine $(n-1)$-space spanned by the face of Δ opposite to a_0. Then $a \in T$ and $x_0 + e_n \in F \subset T$. Hence $y \in T$ where

$$y = a + (x_0 + e_n - a)/t_0 = a_0 + e_n/t_0.$$

Consequently,

$$b(\Delta) \le d(a_0, T) \le |a_0 - y| = 1/t_0 \le p + 1 \le k. \quad ∎$$

5.4. REMARK. One can show that the bounds in 5.3 are sharp. In the rest of the appendix, we do not try to get best possible estimates.

We fix an integer $n \geq 2$ and let $\sigma(\Delta)$ denote the $(n-1)$-dimensional volume of an $(n-1)$-simplex Δ.

5.5. LEMMA: *Let $J \subset \mathbf{R}^n$ be a closed n-interval with edges $t_1, \ldots, t_n$, where $t_j \geq t_n$ for all j. Let $\Delta \subset J$ be an $(n-1)$-simplex. Then*

$$\sigma(\Delta) \leq c_n t_1 \cdots t_{n-1},$$

where c_n depends only on n.

Proof: Assume first that J is a cube with $t_j = t_n$ for all j. Since Δ lies in a ball B of radius $t_n\sqrt{n}/2$, we have $\sigma(\Delta) \leq \sigma(\Delta_1)$, where Δ_1 is the regular $(n-1)$-simplex inscribed to B. Hence $\sigma(\Delta) \leq c_n t_n^{n-1}$ with

$$c_n = \frac{\alpha(n-1)n^{n-1}}{(2n-2)^{(n-1)/2}},$$

where $\alpha(n-1)$ is the volume of the unit $(n-1)$-simplex.

In the general case let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear map

$$fx = (t_1x_1/t_n, \ldots, t_{n-1}x_{n-1}/t_n, x_n),$$

and let Q be the cube $[0, t_n]^n$. Then $fQ = J$. For any $(n-1)$-simplex $A \subset \mathbf{R}^n$ we have $\sigma(fA) \leq t_1 \cdots t_{n-1}\sigma(A)/t_n^{n-1}$. Since $f^{-1}\Delta \subset Q$, the special case gives $\sigma(f^{-1}\Delta) \leq c_n t_n^{n-1}$, and the lemma follows. ∎

5.6. NOTATION. Let $\Delta \subset l_2$ be an n-simplex, and let $\bar{u} = (u(0), \ldots, u(n))$ be a maximal n-sequence in Δ in the sense of 2.4. Then the points $u(j)$ are the vertices of Δ. As before, we set

$$h_j = d(u(j), \operatorname{aff}\{u(0), \ldots, u(j-1)\}), \quad 1 \leq j \leq n.$$

The sequence $\bar{u}$ and the numbers h_j are not uniquely determined by Δ. The volume of Δ is

$$m(\Delta) = h_1 \cdots h_n/n!. \tag{5.a}$$

5.7. THEOREM: *Let $\Delta \subset l_2$ be an n-simplex with maximal n-sequence $\bar{u}$. Then $h_n \le d_n b(\Delta)$, where d_n depends only on n.*

Proof: We may assume that $\bar{u}$ is normalized, that is, $u(0) = 0$ and $u(j) \in \mathbf{R}^j_+$ for $1 \le j \le n$. Then $\Delta \subset \mathbf{R}^n$. Moreover, Δ lies in the n-interval

$$J = [0, h_1] \times [-h_2, h_2] \times \cdots \times [-h_{n-1}, h_{n-1}] \times [0, h_n].$$

Let A be an $(n-1)$-face of Δ with $\sigma(A)$ maximal. Then

$$m(\Delta) = \sigma(A) b(\Delta)/n.$$

Furthermore, since $A \subset J$ and since $h_1 \ge \cdots \ge h_n$, Lemma 5.5 gives

$$\sigma(A) \le 2^{n-2} c_n h_1 \cdots h_{n-1}.$$

By (5.a) we obtain $h_n \le d_n b(\Delta)$ with $d_n = 2^{n-2}(n-1)! c_n$. ∎

5.8. REMARK. The well-known formula

$$\alpha(k) = \frac{\sqrt{k+1}}{2^{k/2} k!}$$

yields the explicit expression $d_n = n^{(2n-1)/2}(n-1)^{(1-n)/2}/2$, which is not the best possible constant.

The following result was needed in 2.10.

5.9. LEMMA: *Suppose that F is an n-dimensional affine subspace of l_2, that $t \ge 0$, that A is a compact subset of $F + \bar{B}(t)$, and that $\bar{u} = (u(0), \ldots, u(n))$ is a maximal n-sequence in A. Let $T\colon l_2 \to l_2$ be an isometry, normalized at $\bar{u}$. Then $TA \subset \mathbf{R}^n + \bar{B}(C_n t)$, where C_n depends only on n.*

Proof: Let $x \in A$. We may assume that $Tx \in \mathbf{R}^{n+1} \setminus \mathbf{R}^n$. Let $\Delta \subset \mathbf{R}^{n+1}$ be the $(n+1)$-simplex with vertices $Tu(0), \ldots, Tu(n), Tx$. Let $P\colon l_2 \to \mathbf{R}^{n+1}$ be the orthogonal projection, and set $F_1 = PTF$. Then $\Delta \subset F_1 + \bar{B}(t)$, and hence $\theta(\Delta) \le 2t$. The lemma follows from 5.3 and 5.7 with $C_n = (n+2)d_{n+1}$. ∎

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